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A. M. Letov

ABSTRACT. Description of certain unsolved problems of automatic control theory, formulated on the basis of the requirements of the Chief Engineer with respect to his product. The problem of synthesizing asymptotically stable systems is discussed, and a description is given of the Chief Engineer's point of view with respect to product design and the role to be played by current theory in this process. A method of solving the problem of synthesizing asympotically stable closed systems is outlined, and a situation leading to the formulation of so-called mixed problems of optimal control theory is described. (A70-32473)

PART I

SYNTHESIS OF ASYMPTOTICALLY STABLE SYSTEMS

§ 1. Foreword

In the past two decades, control theory has made a tremendous step forward /592* in its development. It ceased to be a subject of interest to engineers only. Prominent mathematicians of the century found here a wide area of application for their abstract ideas and currently continued to penetrate into the control field on a heretofore unprecedented scale. On the other hand, the idea of feedback, developed in control theory, upon having been firmly established in technology, penetrated into economics, biology and medicine. This penetration continues also into other areas, particularly into sociology. In view

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* Numbers in the margin indicate the pagination of the original foreign text.

of this, the role and the scientific significance of control theory problems experienced an immense growth.

If we consider these problems separately, i.e., problems of programmed control and problems of dynamics of closed systems with feedback, then we will be forced to conclude that it is justified to express a certain satisfaction only with regards to the solution methods of the former. This satisfaction is predicated on two viewpoints. On one hand, when a system is at the design stage, engineers can clearly conceptualize its purpose and the principal characteristic of the control program: e.g., landing rockets on the moon with minimum fuel consumption. On the other hand, the solution of this type of programming problems is provided by a ready and elegant apparatus of the calculus of variations -- Pontryagin's Maximum Principle and by electronic computers. In this instance, science is capable of offering the Chief Designer a numerical answer to a numerically posed question.

In the dynamics of closed systems, the situation is different. In this instance, science can answer only some of the problems of analysis of the feedback design, suggested by the Chief Designer, and that in the simplest cases only. The most accurately formulated theoretical answers are accompanied by provisos, conditions, and requests for empirically verification. The problems of synthesis of feedback systems, as stipulated by the requirements of the Chief Designer, remain almost untouched.

The purpose of this report is to attract the attention of investigators to problems of synthesis. Those are the problems which are at par with contemporary requirements of technology and contemporary views as to what role must be played by control science in solving the problems that are facing the Chief Designer. I have attempted to attain the most accurate formulations of these problems, keeping in mind that these formulations will not be absorbed by mathematicians precisely as submitted. This is justifiable. The formulations are of a too general nature. More appropriately, these formulations must be construed as an attempt to explain the Chief Designer's

view upon his design in a mathematical form. Mathematicians, if they so desire, will improve these formulations, specifically bring them into a form that will result in solutions. But, in such cases, they will definitely want to also search for solutions. And if this stimulating effect will be accomplished, even if to some extent, the author will consider his attempts justified. /593

§2. Object of Study

The simplest objects of control theory are these objects whose mathematical models can be adequately modelled by ordinary differential equations of the type

$$\dot{x} = f(x, u, t), \quad t \in T. \tag{2.1}$$

Here $x(x_1, \ldots, x_n)$ is the vector of state of the object, $u(u_1, \ldots, u_r)$ is the control vector, t is time determined in the interval $T = [t_0, T]$ or $[t_0, \infty)$. Equation (2.1) is a mathematical rendition (accurate, or sufficiently accurate) of physical (or some other) laws, to which the object is subordinate, and consequently x and u are vectors with real components.

Let N be the domain of action of these laws, i.e., a totality of points x, y, where Equation (2.1) is determined. This may be the entire open domain, or its closed part, defined by inequalities

$$\kappa(x, t) \geq 0, \tag{2.2}$$

$$R(x, u, t) \geq 0. \tag{2.3}$$

The inequalities are either scalar or vector. However, they are such that the freedom of selection of function u and of the motion of the image point x are still retained. Insofar as functions κ,R are concerned, it is usually assumed that function κ is differentiable the required number of times, and

the matrix $\left\| \frac{\partial R}{\partial u} \right\|$ exists in domain N and has a maximum rank, that equals the dimension of vector R.

Definition 1. Select a certain class of function u (e.g., piecewise continuous functions) and assume that the control of u is admissible, provided $u \in U$ takes on numerical values from domain N for given x.

Definition 2. Any solution of Equation (2.1) from domain N, determined with any admissible u, is said to be an admissible trajectory.

In order to be certain of the existence of such a trajectory, we will assume that the vector function f is smooth everywhere in domain N, and that the system (2.1) is controllable. Understandably, the requirement as to the smoothness of f can be weakened.

Definition 3. Equation (2.1), which satisfies the requirement of this section, is said to be the mathematical model of the control object. Insofar as Equation (2.1) is concerned, we can formulate a multitude of different problems which may well constitute the subject of science now known as the theory of control.

§3. A Quick Look in Retrospect

It is easy to look in the past because it is not that far removed. Also, because a quick look is not sufficiently deep. Nonethless, this viewpoint is expedient, particularly for the purposes of this paper.

In the past, control theory has been the theory of automatic regulation. /594 In rough terms, the latter was reduced to the following. Primarily, control objects were considered whose characteristics did not change in time; as a consequence, Equation (2.1) did not contain t explicitly. The possible stabilized states of x* were studied as solutions of the equation

$$f(x, u) = 0 \tag{3.1}$$

and the range of possible stabilized deviations of control elements $u_1 \leqslant u^* \leqslant u_2$, within which it was possible to maintain the state of x* was determined. This happened to the first point of the theory, i.e., the statics of regulation.

Subsequently, an equation for the increments was constructed. It was assumed that the actual state of object x would be close to x*, and that the actual position of the control element u would be close to u*. It was further assumed that

$$x = x^* + \beta,$$
$$u = u^* + \xi. \tag{3.2}$$

The equation for the increments has the form

$$\dot{y} = Ay + b\xi + \varphi(y, \xi), \tag{3.3}$$

where A, b are constant matrices, constructed in a certain manner for a possible mode of state of x*, u*. In this framework, no reference was made to Lyapunov [1]. Pioneers of the regulation theory [2] apparently acted in this direction guided by their intuition and their genius, even though when their contributions were made public, there did indeed exist a finished and developed strict conception of perturbed-unperturbed motion.

The second milestone in the theory of automatic control was interest in problems of dynamics, related to Equation (3.3). This equation defines what is known in control as a transitional process; in mechanics it is known as the perturbed Lyapunov motion.

The class of considered dynamic problems actually involved analytical questions. Specifically, an engineer set some function ξ as a known function of y and of some vector parameter $p(p_1, \ldots, p_l)$

$$\xi = \xi(y, p); \tag{3.4}$$

This function is said to be the law of control. Substituting it into (3.3) yields a closed system

$$\dot{y} = Ay + b\xi(y, p) + \varphi(y, \xi(y, p)). \tag{3.5}$$

with respect to which, different problems of analysis were formulated, namely, the stability problem and the regulation of quality problems. These problems related to a choice of selecting some vector p. The term "regulation" in this context meant nothing else but the required maintenance of the stationary state of object x*, which was eventually reached at constant deviation of "control" u*.

Subsequent technological develpment gave rise to the emergence of much more complex problems. In these, the characteristics of the object became time dependent t themselves, moreover, x*, u* defined a priori the generally speaking unstabilized motion which was so selected as to precisely or adequately satisfy condition

$$\dot{x}^* = f(x^*, u^*, t). \tag{3.6}$$

In this instance, (3.3) and the control law per se was rendered as

$$\dot{y} = A(t)\, y + b(t)\, \xi + \varphi(y, \xi, t), \tag{3.7}$$

$$\xi = \xi(y, p). \tag{3.8}$$

It would be natural to assume that now the vector parameter p may depend upon t. The search for it is also included in some analytical problems. And even though the sclutions of these problems leave room for improvement, they are an extremely useful mathematical tool, which leads to obtaining answers to many problems of first-rate practical significance and, at the present time, their solution methods are firmly embedded in engineering practices. The purely scientific and rather valuable significance of analytical problems is

that they include the basic concepts which express the most important and practically acceptable properties of closed systems. However, it became obvious that a drawback of analytical problems is the narrowness of their mathematical formulation. In the optimal case, the theory can stipulate whether or not the feedback diagram, suggested by the Chief Engineer (3.8), is reflective of his design concepts, and how its parameter should be selected. So far, control theory cannot offer the Chief Engineer a selection of a multitude of feedback diagrams for his final choice. For this reason, a natural and necessary transition to problems of a new type is in order -- namely, problems of synthesis.

The attempts at synthesis, undertaken here, are not entirely new. On a purely mathematical plane, they were preceded by the ideas and investigations of N. P. Yerugin [3], as well as other authors [4, 5] dealing with the construction of differential equations with preset properties. On the other hand, there has been an experiment on developing a solution of one of the partial problems of this plan -- namely, the problem of "analytical construction of optimal regulators" [6]. The practical advantage of the problem is that it can be effectively solved in a large number of cases which have considerable applied significance. Of great interest for practical purposes is also the experience in solving analytical and synthesizing problems, described in handbooks [7, 8]. A new approach to the problem is aimed at providing a basis for regulator design from the viewpoint of the Chief Designer of the product (2.1). This viewpoint of the Chief Designer we will attempt to describe in the second part of this report. Let it be said here that, regardless of this viewpoint, the Chief Designer always deals with closed systems which, in Lyapunov's sense are asymptotically stable. This is a fundamental position, that cannot be escaped. For this reason, it is above all necessary to find the control laws which impart to a closed system the property of asymptotic stability. Some of these laws are known to the Chief Designer empirically. However, the Chief Designer must know as many as possible of such laws, or all of them.

The most important part of constructing the theory is to provide this extremely valuable information to the Chief Designer.

§4. Synthesis of Asymptotically Stable Systems

Consider the equation of perturbed motion

$$\dot{y} = F(y, \xi, t) + f^*(t), \tag{4.1}$$

where f* is the perturbing force. Its origin could be f* ≡ 0., if (3.6) is satisfied precisely. If (3.6) is satisfied approximately, then either on account of the approximate nature of the programmed control u*, or in view of the emergence of previously unconsidered forces in (3.6), the right half of (3.7) will contain an additional term f*(t). Equation (4.1) has the property $F(0,0,t) \equiv 0$.

In this paper, we will only consider the case when $f^*(t) \equiv 0$. We will adopt the hypothesis that if (3.1) is controllable, then (4.1) is also controllable.

Definition 4. Assume that System (4.1) is observable within the coordinates $y_1,\ldots y_p (p \leq n)$, provided the Chief Designer has the technical means available to him to measure components $y_1,\ldots y_p$ of vector y at any given moment of time t. For example, an ordinary aircraft in longitudinal motion is not a completely observable system because there exists (as yet) no device which can measure its angle of attack. Naturally, Definition 4 is arbitrary, because one could pose the problem of designing a device which would measure the angle of attack, or use the solution of the problem on observability, as suggested in [9],. However, the Chief Designer is not justified to take this step involving a deliberate complication of the control equipment since he is not convinced that this is really necessary.

Definition 5. A totatility of coordinates $y_1,\ldots y_p \in N$ of the observation is said to be the control field [10] and is denoted by symbol P. When p = n, such a control field is congruent with the range of all values of y in domain N. Such a field is referred to as the maximal field[1].

Consider a set M of all admissible functions

$$\xi = \xi(y_1, \ldots, y_p, \eta). \tag{4.2}$$

determined in the field P.

Definition 6. The substitution of ξ in Equation (4.1) by any admissible Function (4.2) is said to be the closure of the system, and a system so constructed is said to be a closed system. It is defined by equation

$$\dot{y} = F(y, \xi(y_1, \ldots, y_p), \eta, t). \tag{4.3}$$

Definition 7. Let any admissible Function (4.2), determined in the control field P, belong to the totality of feed-back laws, provided that its closure of System (4.1) makes the undisturbed state y = 0 of the latter asymptotically stable in the Lyapunov sense.

Problem 1. Given mathematical Model (4.1) of the control object and field P. Determine the totalility S_p of feedback laws (4.2) determined over field P.

Consider, as an example, a system which consists of a neutral aircraft with an automatic pilot [11]

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[1] In control practice, there exist supermaximal fields which include the measurement of several values $y_1,\ldots. y_n$. Obviously, if a control field is supermaximal, then it can also become maximal.

$$T^2\ddot{\varphi} + U\dot{\varphi} + \mu = 0,$$

$$\dot{\mu} = f(\sigma),\ \sigma = a\varphi + E\dot{\varphi} + G^2\ddot{\varphi} - r\mu - R\dot{\mu},$$

where $f(\sigma)$ is a nonlinear limited function with the property $f(0) = 0$, $\sigma f(\sigma) > 0,\ \sigma \neq 0$.

Here, the control signals $a\varphi, E\dot{\varphi}, G^2\ddot{\varphi}$ are emitted by the respective gyroscopes, whereas signal $r\mu + R\dot{\mu}$ is generated by the feedback element. A totality of coordinates $y_1 = \varphi,\ y_2 = \dot{\varphi},\ y_3 = \mu$ is used as the control field, inasmuch as the initial system is equivalent to another system:

$$\dot{y}_1 = y_2, \qquad \dot{y}_2 = -\frac{U}{T^2}y_2 - \frac{1}{T^2}\mu,$$

$$\dot{\mu} = f\left[a\varphi + \left(E - U\frac{G^2}{T^2}\right)\dot{\varphi} - \left(r + \frac{G^2}{T^2}\right)\mu - R\dot{\mu}\right],$$

where the effect of $G^2\ddot{\varphi}$ is easilly taken into account by an appropriate selection of the amplification coefficient of signals whihh are proportional to φ, μ . There exists, however, a field of minimum measurements. To find this field, it is sufficient to construct a characteristic polynomial of the system for the case when it is linear $f(\sigma) = h\sigma$. This polynomial is a Hurwitz polynomial if and only if

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$$U + (rT^2 + G^2)h^* > 0,$$

$$E + rU > 0,$$

$$a > 0,$$

$$[U + (rT^2 + G^2)h^*][rU + E] > aT^2,\ h^* = \frac{h}{1 + hR} > 0.$$

The field of the second measurement is minimal, and includes either observations of coordinates, φ, μ, or of coordinates $\varphi, \dot{\varphi}$. The same field remains minimal alsc if $f(\sigma)$ is nonlinear. Minimal fields are of special interest to the Chief Designer, because they satisfy his requirements for simplicity and design reliability as well as for a decrease of weight and dimensions.

Definition 8. The field P of minimal measurement or which at least one feedback law (4.2) exists, is said to be the minimal field.

Problem 2. Of all feedback laws, which correspond to Problem 1, find those which are determined above the field of minimal measurement.

§5. Range of Atraction

Definition 9. Consider Equation (4.3). Let $\Sigma(y_1, \ldots, y_n)$ be any of the synthesized feedback laws. The set Σ of points y_0, containing the origin of coordinates for which the solution $y = 0$ of Equation (4.3) retains its properties of asymptotic stability, is said to be the range of attraction of solution $y = 0$ [12 - 14].

Problem 3. Assume a mathematical model of a control object is given and the entire set of feedback laws is determined. Plot an attraction range of the solution $y \equiv 0$ for any case of closure. The practical significance of the range of attraction in synthesis problems is enormous. It consists in that in any case when there is the initial deviation y_0 in a given closed system, where this deviation does not belong to that range, the control will not be able to handle its problem which will, in turn, result in the destruction of the closed system.

It follows that the range of attraction is a range that provides life for the closed system, and the Chief Designer is entitled to know the boundaries of this range and is also entitled to request its definition.

Until now, theory did not sufficiently emphasize the development of effective determination methods for the boundaries of the attraction range.

PART II /598

SYNTHESIS OF SYSTEMS WITH A SET QUALITY

§6. The Viewpoint of the Chief Designer of the Product

Frequently, the Chief Designer does not require an absolutely optimal regulator . The Chief will be quite satisfied with a control conforming to his technical specifications. These specifications, in addition to calling for simplicity, reliability and economy, include the requirements for the dynamics of the transitional process in a closed system. The importance of these requirements cannot be overestimated. They are predicted on experience, common sense, and always have a specific nature determined by the nature of the controlled object. For example, for a nuclear reactor, the transitional process must attenuate within 4 to 5 seconds, and its maximum deviation from a stabilized state must not exceed known boundaries of the domain N. A violation of any one of these requirements will result in a nuclear explosion. It should be assumed that, all other conditions being equal, the three primary requirements of the Chief Designer -- simplicity, reliability and economy of the control -- coincide with the feedback laws, determined in minimal fields.

We will now attempt to formulate an approach for satisfying the requirements that relate to the dynamics of a transitional process in a closed system. As pointed out above, these requirements were formulated in the practice and in the theory of control and were discussed in the literature, dealing with solution of analytical problems. In this frame of reference, of special importance is the report by N. G. Chetayev 'O vylore parametrov ustoychivosti mekhanicheskikh sistem (On the Selection of Stability Parameters of Mechanical Systems), PMM (Prikladnaya mekhanika i matematika), Vol. 15, No. 2". This report was the first to offer a correct treatment and solution of the speed of response problem.

§7. The Speed of Response Problem

If we could easily solve the synthesis of control problems which are optimal with respect to the speed of response, even then the following problem formulation would make sense.

Consider two spheres $\|y\| = R$, $y_0 = y(t_0)$, $\|y(t^*)\| = \varepsilon$ where $R > \varepsilon$ are given integers.

Consider now the set (4.2) of all feedback laws. Take any one of them and perform the closure. Let us obtain Equation (4.3). Consider now the entire set of solutions $y = y(t, t_0, y_0)$ of Equation (4.3) which begin on sphere R. We will refer to them as R-solutions. Inasmuch as System (4.3) is asymptotically stable for any y_0 on sphere R, there exists a moment of time t_0^* for which conditions

$$\begin{aligned} \|y(t_0^*, t_0, y_0)\| &= \varepsilon, \\ \|y(t, t_0, y_0)\| &< \varepsilon \end{aligned} \tag{7.1}$$

hold at any $t > t_0^*$.

Denote $t^* = \sup_{y_0} t_0^*$. From the definition of t_0^* it follows that t^* exists.

Definition 10. The interval $t^* - t_0$ is said to be the control time (ot the attenuation of the transitional process) in a closed system (4.3), if any R-solution reaches sphere ε when $t_0 < t^*$ and remains inside of it when $t > t^*$. Obviously, the control time is a functional of the type $t^*(R, \varepsilon)$. Let T be a given integer.

Problem 4. A set M of feedback laws (4.2) is given. It is required to separate a subset M_1 of this set in which the control time in the closed system will satisfy condition

$$r^*(R, \varepsilon, \Xi) < T. \tag{7.2}$$

We will refer to this problem as the synthesis problem of high speed of response regulators.

§8. The Overshoot Problem

Definition 11. Consider any R-solution of Equation (4.3), determined for any feedback law, and any function $\Phi(y)$ which assumes positive values. Examples of such functions are $\Phi = \|y\|^2$, $\Phi = \sum a_i |y_i|$, $a_i > 0$, etc. The ratio

$$\sigma = \sup_t \frac{\Phi(y) - S}{S}, \quad S > 0, \tag{8.1}$$

is said to be the overshoot quantity. Obviously, if σ exists, then it must be positive. If, however, for some ξ, $\sigma < 0$, then we can stipulate that system (4.3) has no overshoot. Obviously, $\sigma = \sigma(R, S, \Xi)$ is a functional. Let $\bar{\sigma}$ be the given integer.

Problem 5. Let M be a set of feedback laws (4.2). Determine a subset of it M_2 on which

$$\sigma(R, S, \Xi) < \bar{\sigma}. \tag{8.2}$$

§9. The Monotonic Attenuation Problem

Definition 12. Let α be a given positive integer. The transitional process in System (4.3) is said to be monotonically attenuating if there exists a feedback law (4.2) such that any R-solution of Equation (4.3) satisfies condition

$$\|y(t, t_0, x_0)\| \leq A e^{-\alpha(t - t_0)}, \quad A, \alpha > 0. \tag{9.1}$$

Otherwise, the condition of monotonic attenuation (9.1) is replaced by condition

$$\dot{s}(t, s) < 0. \tag{9.2}$$

Problem 6. Given set M of feedback laws; determine a subset of it M_3 on on which any R-solution of Equation (4.3) will attenuate monotonically.

§10. Terminal Control

Definition 13. We shall assume that we are dealing with a terminal control problem when the state of the closed System (4.3) that interests us, at a certain given moment of time T, is completely defined by a given Meyer functional of the type

$$\Phi = \Phi(y(T), T), \tag{10.1}$$

which does not take on negative values in the range of variation of its arguments. The problem of an automatic landing of an aircraft is a classic example of a terminal control problem. When only longitudinal motion is involved, this state is completely defined by the altitude of the aircraft's center of gravity above the landing strip and by the vertical speed of moment T of the completion of the maneuver referred to as compensation figure. /600

The quality problem of some industrial products is also a terminal control problem.

Problem 7. Given set M of feedback laws. Determine a subset of it M_4 on which the value of Functional (10.1), computed along any arbitrary R-solution, does not exceed the given integer $\bar{\Phi}$:

$$\Phi(y(T), T) < \bar{\Phi}. \tag{10.2}$$

§11. Autonomy.

Definition 14. System (4.3) is said to be autonomous within a group of coordinates $y_1,\ldots,y_r$ if its structure is such that solutions $y_1(t,t_0,y_0),\ldots$ $y_r(t,t_0,y_0),r<n$ depend only on the initial values of $y_{10},\ldots, y_{r0}$, and do not depend upon the initial values of other coordinates.

Definition 15. Let ε be an arbitrarily small positive integer. System (4.3) is defined as autonomous with an accuracy of up to ε, within a group of coordinates $y_1,\ldots,y_r$, if with initial values $y_{10} = \ldots y_{r0} = 0$ and arbitrary values of $y_{r+1.0},\ldots y_{n0}$, taken over a sphere R, for the solutions of

$$y_1(t, t_0, 0, \ldots, 0, y_{r+1,0}, \ldots, y_{n0}), \ldots, y_r(t, t_0, 0, \ldots, 0, y_{r+1,0}, \ldots, y_{n0})$$

the inequality

$$y_1^2 + \ldots + y_r^2 < \varepsilon^2 \tag{11.1}$$

holds with any $t \geq t_0$. When $\varepsilon = 0$, definitions 14 and 15 are equivalent.

Problem 8. Given a set of M feedback laws. Determine a subset M_5 of this set, on which the System (4.3) is autonomous (or autonomous with an accuracy of up to ε) with respect to the group of coordinates $y_1, \ldots, y_r$, $r<n$.

§ 12. Invariance

Consider equation

$$\dot{y} = F(y, t(y_1, \ldots y_n, 0, t) + f(t) \tag{12.1}$$

which is closed by one of the feedback laws, belonging to the M-set. One of the principal purposes of the law $t \in M$ is the opposition to the perturbing

forces that are defined by the vector function f*. Here, we will elaborate on two approaches, to the solution of the opposition problem. We will begin with the approach that was initially developed by G. V.Shchipanov.

Let α,β be fixed integers $(\alpha, \beta = 1, \ldots, n)$. We stipulate that the System (12.1) is invariant along coordinate y_α with respect to perturbing force $f_\beta(t)$, if a feedback law $u \in M$ can be found with which the solution y_α of Equation (12.1) does not depend upon $f_\beta(t)$.

Let ε be an arbitrarily small positive integer. Denote the solution of Equation (12.1), when $u_\alpha(t)$ by $f_\beta(t) \equiv 0$. System (12.1) is said to be invariant with an accuracy of up to ε over coordinate y_α with respect to the perturbing force $f_\beta(t)$, if a feedback law $u \in M$ can be found for which the solution y_α satisfies the inequality

$$|y_\alpha(t, t_0, y_0) - y_\alpha^0(t, t_0, y_0)| < \varepsilon. \tag{12.2}$$

Problem 9. A set M of feedback laws is given. Determine a subset M_6 of this set on which System (4.3) is invariant (this invariance having an accuracy of up to ε) along coordinate y_α with respect to the perturbing force $f_\beta(t)$.

Another approach to opposition is related to the concept of technical stability, introduced originally by N. D. Moiseyev. Assume that f*(t) is an arbitrary function that satisfies the condition

$$|f^*(t)| < \eta \tag{12.3}$$

where η is the given u, and possibly a sufficiently small integer. If $u \in M$, it follows that System (12.2) is stable with constantly acting forces [15]. This means, that in the case that is important for practical purposes, when $y_0 = 0$ and the disturbing force $f^*(t)$ begin to act, the system leaves the

equilibrium position y = 0. However, its deviation will be restricted by a certain hypersphere with radius $r(\eta, \xi)$ which in turn is a certain functional of ξ. We will refer to it as the "excitation sphere of a closed system". Let $\bar{r}$ be the given integer.

Problem 10. Given a set of M feedback laws. Determine a subset M_7 of this set on which the value of the functional $r(\eta, \xi)$, which defines the excitation sphere of a closed system, satisfies the condition

$$r(\eta, \xi) \leqslant \bar{r}. \tag{12.4}$$

Condition (12.4) can also be referred to as the comfort criterion of control.

§13. The Intersection Principle.

The formulation of other quality criteria of transitional processes in closed systems could be continued (e.g., the criterion of minimal complexity, [8]). We will now discuss this. Assume now that some set M of feedback laws (4.2) is known. Assume further, that as a result of selection of the laws of this set, in accordance with criteria (7.2), (8.2), (9.1) and others, it became possible to separate subsets $M_1, \ldots\ldots M_7$, and that only one quality criterion is satisfied on one each of these subsets. Obviously, if we wish that the designed system satisfy the entire totality of the formulated criteria, then the sets M, $M_1, \ldots, M_7$ must have a nonempty intersection. This is essentially the substance of the intersection principle. The role of minimal fields becomes especially significant in view of this principle. Assume now that a set M' of feedback laws, determined on the minimal field P_{min}, is known for the given control object. Assume further, that in addition to the requirement of asymptotic stability of the closed system we also want to satisfy the speed of response criterion. The Chief Designer, obviously, will only approve the dimensional increase of the control field if the speed of response criterion cannot be satisfied on the P_{min} field.

PART III

§14. One of the Methods of Synthesis of Feedback Laws

One common method of seeking solutions in the class of sufficient solutions should be pointed out. We will illustrate this method on an example.

Assume that ξ is a scalar, and that Equation (4.3) has the form

$$\dot{y} = g(y, t) + m(y, t)\xi. \tag{14.1}$$

Assume further that when the regulator is turned off ($\xi = 0$), the control object

$$\dot{y} = g(y, t) \tag{14.2}$$

is stable with respect to unperturbed motion

$$y = 0. \tag{14.3}$$

Assume that the admissible controls constitute a set of piecewise continuous functions which assume their values from the domain

$$|\xi| < \xi \tag{14.4}$$

Assume further that $V_1(y, t)$ is a Lyapunov function, which reveals the stability of the solution of (14.3). This means that its complete derivative

$$\frac{dV_1}{dt} = \frac{\partial V_1}{\partial t} + \frac{\partial V_1}{\partial y} g(y, t) = -\omega(y, t), \tag{14.5}$$

computed on the trajectories of Equation (14.2) is a fixed sign function (or a constant term function) of a sign that is opposite to the sign of V_{1*}.

On the strength of Equation (14.1), we obtain

$$\frac{dV_1}{dt} = -\alpha(y, \eta) + \frac{\partial V_1}{\partial y} \cdot m(y, \eta, \xi). \tag{14.6}$$

Formula (14.6) permits us to identify a set M of admissible controls of ξ, which guarantee the stability of System (14.1). For example, we have

$$\xi = -\Gamma \operatorname{sign} \sigma, \quad \sigma = \frac{\partial V_1}{\partial y} \cdot m(y, \eta). \tag{14.7}$$

The problem of the asymptotic stability of a closed system is resolved by function

$$\alpha(y, \eta) + \Gamma \left| \frac{\partial V_1}{\partial y} \cdot m(y, \eta) \right| = U(y, \eta). \tag{14.8}$$

In the instance of stabilized motion, it is advisable to use Lyapunov's strong theorem of asymptotic stability [12, 13, 15]. This theorem can be formulated as follows:

Consider the closed system

$$\dot{y} = \varphi(t, y(t), \ldots, y_n). \tag{14.9}$$

For the state

$$y = 0 \tag{14.10}$$

to be asymptotically stable, it is sufficient that there exists a fixed-sign positive function V(y); its complete derivative, in view of equations (14.9), is a fixed-sign negative function U(y); however, this function is such that on the manifold

$$U(y) = 0 \tag{14.11}$$

there is no single whole trajectory of System (14.9), other than y = 0.

This theorem can also be useful in answering the second question: Is the field $P(y_1,\ldots y_p)$ minimal in the class of sufficient solutions of the synthesis problem. Obviously, the set M contains also equation

$$\xi = -\mathrm{sat}\,\rho\sigma_1,\ \rho > 0, \tag{14.12}$$

Here, the function sat signifies /603

$$\mathrm{sat}\,\rho\sigma_1 = \begin{cases} \bar{\xi}, & \rho\sigma_1 > \bar{\xi} \\ \rho\sigma_1, & \rho|\sigma_1| \leq \bar{\xi} \\ -\bar{\xi}, & \rho\sigma_1 < -\bar{\xi} \end{cases} \tag{14.13}$$

with ρ being an arbitrary positive integer. A more complex feedback law is determined by the formula

$$\xi = -\varphi(y, i)\,\mathrm{sign}\,\sigma_1(y, i), \tag{14.14}$$

where $\varphi(y, i)$ is an arbitrary piecewise continuous positive function of y, i, which belongs to the class of piecewise continuous functions; it is important in the domain

$$|\varphi| < \bar{\xi}. \tag{14.15}$$

In another case, if $\varphi(y, i)$ is a function that satisfies condition $\varphi(y, i) \geq \rho > 0$, we can set

$$\xi = -\mathrm{sat}\,\varphi\sigma_1, \tag{14.16}$$

The latter means that for ξ we can take any function

$$\xi = -f(\sigma_1),\ |f(\sigma_1)| \leqslant \bar{\xi},\ f(0) = 0,\ \sigma_1 f(\sigma_1) > 0,\ \sigma \neq 0.$$

Subsequent and almost unrestricted expansion of the set is accomplished by using new Lyapunov functions.

Let $V_2(y, t)$ be another function, having the same properties as function V_1. Then it will obviously follow that

$$\xi = -\bar{\xi}\operatorname{sign}\sigma_2,\ \xi = -\operatorname{sat}\rho\sigma_2,\ \rho \geqslant 0,$$
$$\sigma_2 = \frac{\partial V_2}{\partial y} \cdot m(y, t). \tag{14.17}$$

Obviously, if V_1 and V_2 are two Lyapunov functions, then their sum and products will also be Lyapunov functions. Hence, the controls

$$\xi = -\bar{\xi}\operatorname{sign}(\sigma_1 + \sigma_2),\ \xi = -\operatorname{sat}\rho(\sigma_1 + \sigma_2), \tag{14.18}$$

as well as controls

$$\xi = -\bar{\xi}\operatorname{sign}(V_1\sigma_1 + V_2\sigma_2),\ \xi = -\operatorname{sat}\rho(V_1\sigma_1 + V_2\sigma_2) \tag{14.19}$$

will also be feedback laws.

The methods of expanding set M is obvious: each new Lyapunov functions makes it possible to form new controls either using Formulas (14.7), (14.12), or using Formulas (14.18), (14.19). The possibilities that open up when these methods are pursued are illustrated in the following examples.

§15. Satellite Stabilization in a Given Direction. The Case of Small Initial Disturbances.

Consider a control object defined by Equations in [16]

$$\begin{aligned}
\dot{y}_1 &= y_2,\\
\dot{y}_2 &= \frac{A-B}{C}\,\bar{\omega}^2 y_1 + \frac{B-A-C}{C}\,\bar{\omega} y_4 + u_2,\\
\dot{y}_3 &= y_4,\\
\dot{y}_4 &= -\frac{B-A-C}{A}\,\bar{\omega} y_2 - (3\bar{\omega}^2 + \bar{\omega}^2)\,\frac{B-C}{A}\,y_3 + u_4.
\end{aligned} \tag{15.1}$$

We are considering small oscillations of the satellite about the equilibrium position y = 0. The satellite moves at an angular velocity $\bar{\omega}$ along a circular orbit with radius R. Here, A. B C. are the principal central moments of inertia of the satellite; $\alpha_1 = \alpha'$ is the angle of the satellite with respect to the main connecting axis; $z', \alpha_3 = \gamma'$ is the angle of axis z' in a direction that is perpendicular to the plane of the orbit. For determinancy, it is assumed that $B > A, B > C$. The transformation

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$$\begin{aligned}
&x_i = \alpha_i x_i \quad (i = 1, 2, 3, 4),\\
&\alpha_1 = 1,\ \alpha_2 = \bar{\omega}\sqrt{\frac{B-A}{C}},\ \alpha_3 = \frac{1}{2}\sqrt{\frac{B-A}{B-C}},\\
&\alpha_4 = \bar{\omega}\sqrt{\frac{B-A}{A}}
\end{aligned} \tag{15.2}$$

reduces Equation (15.1) to the form

$$\begin{aligned}
\dot{x}_1 &= \alpha_2 x_2,\\
\dot{x}_2 &= -\alpha_3 x_1 + \alpha_4 x_3 + u_2,\\
\dot{x}_3 &= \alpha_4 x_4,\\
\dot{x}_4 &= -\alpha_3 x_2 - \alpha_2 x_3 + u_4,
\end{aligned} \tag{51.3}$$

$$\alpha_2 = \bar{\omega}\sqrt{\frac{B-A}{C}},\quad \alpha_3 = \frac{B-A-C}{\sqrt{AC}}\,\bar{\omega},\quad \alpha_4 = \bar{\omega}\sqrt{\frac{B-C}{A}}.$$

We will explore the problem for two cases of determining the control.

Assume that we want to synthesize admissible controls with feedback, determined over the field $P(y_2, y_4) = P(z_2, z_4)$, satisfying the condition

$$|u_2| < \bar{u}_2, \quad |u_4| < \bar{u}_4 \tag{15.4}$$

where $\bar{u}_2$ and $\bar{u}_4$ are given integers.

We will seek the controls in the class of sufficient stability conditions, guaranteed by the Lyapunov function

$$V = \tfrac{1}{2}z^2. \tag{15.5}$$

The complete derivative function on the trajectories of System (15.3) is

$$\dot{V} = u_2 \frac{\partial V}{\partial z_2} + u_4 \frac{\partial V}{\partial z_4}. \tag{15.6}$$

The synthesized equations have the form

$$u_2 = -\bar{u}_2 \operatorname{sign} z_2, \quad u_4 = -\bar{u}_4 \operatorname{sign} z_4. \tag{15.7}$$

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Actually, from (15.6), if follows that

$$\dot{V} = -\bar{u}_2 |z_2| - \bar{u}_4 |z_4|. \tag{15.8}$$

Even though $\dot{V}$ is not a fixed-sign negative function, it can be easily established that the manifold

$$\bar{u}_2 |z_2| + \bar{u}_4 |z_4| = 0 \tag{15.9}$$

does not contain whole trajectories of the closed system, other than $z = 0$, Actually, Condition (15.9) is satisfied only when $z_3 = z_4 = z_0$. But then $u_2 = u_4 = z_1 = z_3 = 0$. It follows that the control of (15.7) guarantees asymptotic stability to the closed system. It can be easily verified that any other admissible control of the form

$$u_3 = -\operatorname{sat}\rho z_3,$$
$$u_4 = -\operatorname{sat}\rho z_4 \qquad (15.10)$$

has the same property; here ρ is an arbitrary positive integer. To prove that, it is sufficient to construct a characteristic equation of the system in a linear zone. It has the form

$$\begin{vmatrix} -\lambda & m & 0 & 0 \\ -m & -\lambda-\rho & 0 & n \\ 0 & 0 & -\lambda & k \\ 0 & -n & -k & -\lambda-\rho \end{vmatrix} = 0 \qquad (15.11)$$

or in ordinary form

$$\lambda^4 + 2\rho\lambda^3 + (\rho^2 + n^2 + k^2 + m^2)\lambda^2 + \rho(k^2 + m^2)\lambda + k^2m^2 = 0. \qquad (15.12)$$

It can be easily demonstrated that the left half of Equation (15.12) contains a Hurwitz polynomial. It is of interest to note that the control field in the case of Controls (15.7), (15.10) is not minimal. A minimal field is either the field $P(y_2)$, or field $P(y_4)$. We will check this. Consider controls of (15.7). In the event when $u_4 = 0$, we have $\dot{V} = -\bar{u}_4|z_4|$, and $\dot{V}$ vanishes on the manifold $z_2 = 0$. But then $uz_4 = mz_1$. Inasmuch as $z_1 = \text{constant}$, it follows that $z_4 = \text{constant}$. Consequently, $\dot{z}_3 = \text{constant}$, and hence $z_4 = z_1 = 0$. In the case $u_2 = 0$, it follows that $\dot{V} = -\bar{u}_4|z_4|$ and $\dot{V}$ vanishes when $z_4 = 0$. But then, $kz_4 = -nz_2$, whereby $\dot{z}_3 = \text{constant}$. It follows that $z_1 = z_2 = z_3 = 0$. In all other instances, $\dot{V}$ is strongly negative and decreases monotonically. To control (15.10), we have polynomial

$$\lambda^4 + \rho\lambda^3 + (n^2 + k^2 + m^2)\lambda^2 + \rho k^2\lambda + k^2m^2 = 0 \qquad (15.13)$$

when $u_4 = 0$, and the polynomial

$$\lambda^4 + \rho\lambda^3 + (n^2 + k^2 + m^2)\lambda^2 + \rho m^2\lambda + k^2m^2 = 0 \qquad (15.14)$$

with $u_2 = 0$. Both polynomials are Hurwitz polynomials. If the admissible control equations are determined by condition

$$u_3^2 + u_4^2 \leqslant \bar{u}^2, \tag{15.15}$$

then, with the same Lyapunov function, we must set

$$u_3 = -\rho z_3, \quad u_4 = -\rho z_4. \tag{15.16}$$

For this reason,

$$\dot{V} = -\rho(z_3^2 + z_4^2). \tag{15.17}$$

The positive multiplier ρ must be selected from the condition /606

$$\rho \leqslant \frac{\bar{u}}{\sqrt{z_3^2 + z_4^2}}. \tag{15.18}$$

whence, we find that

$$\dot{V} = -\bar{u}\sqrt{z_3^2 + z_4^2}, \tag{15.19}$$

$$u_3 = -\frac{\bar{u}z_3}{\sqrt{z_3^2 + z_4^2}}, \quad u_4 = -\frac{\bar{u}z_4}{\sqrt{z_3^2 + z_4^2}}. \tag{15.20}$$

The control field, defined by controls (15.20), is not minimal either. The latter is established by the same method as previously.

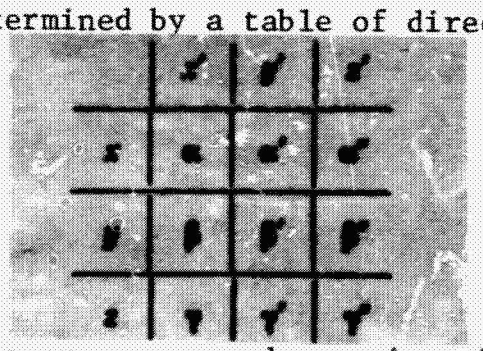
The range of attraction of point $z = 0$ is the entire domain z, in which Equations (15.1) retain their validity.

§16. Satellite Stabilization in a Given Direction: The Case of Arbitrary Initial Perturbations.

We will adopt the equations of motion of the satellite from [16]. Let CXYZ be an immobile right trihedron with an origin at the center of the Earth, the axis CY, coinciding with the axis of the Earth, the axis CZ aimed at the point of the vernal equinox.

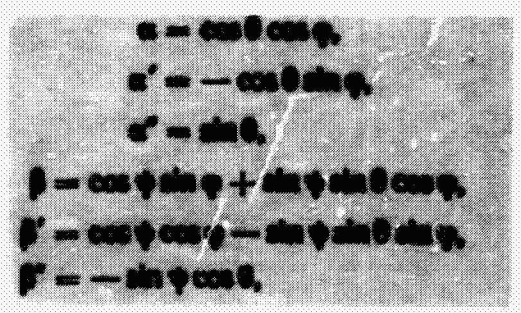
Let oxyz be on orbital right trihedron with the ox-axis directed along the tangents to the orbit, in the direction of the motion of the satellite center of gravity 0, whereas axis oz is directed along the radius vector R of point 0.

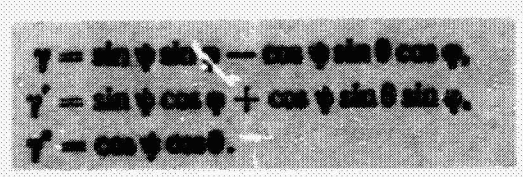
Let ox'y'z' be the principal central right trihedron, rigidly connected with the satellite's body. The satellite position, with reference to the orbital trihedron, is determined by a table of directional cosines.



(16.1)

In this model, all cosines are expressed unequivocally by means of 3 angles, φ, θ, ψ using standard formulae





(16.2)

Here, θ is the angle between the axis oz' of the satellite and its projection upon plane yz; ψ is the angle between that projection and axis oz; ϕ is the turn of the satellite about axis z' (Figure 1). All turns are right turns. /607

The complete system of equations consists of equations predicated upon the theorem of momentum:

$$A\dot{p} = (B-C)qr + 3\frac{\mu}{R^3}(C-B)\gamma'\gamma'' + L_{x'},$$

$$B\dot{q} = (C-A)rp + 3\frac{\mu}{R^3}(A-C)\gamma''\gamma + L_{y'},$$

$$C\dot{r} = (A-B)pq + 3\frac{\mu}{R^3}(B-A)\gamma\gamma' + L_{z'},$$

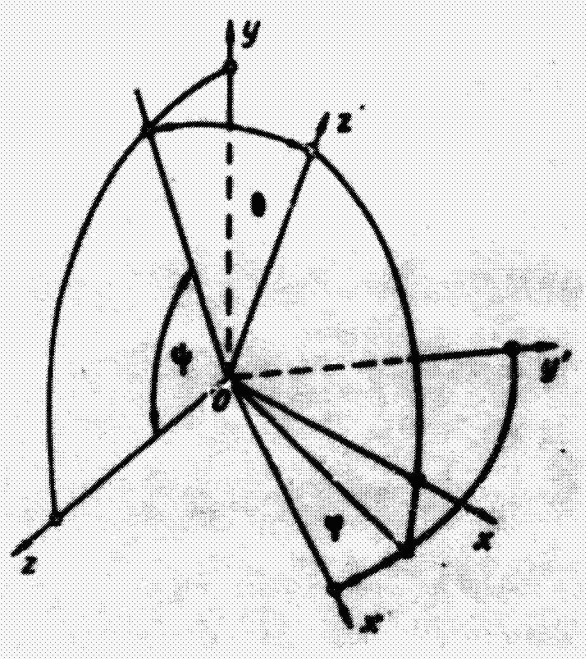


Figure 1. Coordinate axis.

as well as kinematic equations for directional cosines

$$\begin{aligned}
\dot{\alpha} &= \alpha' r - \alpha'' q - \omega\gamma, \\
\dot{\alpha}' &= \alpha'' p - \omega\alpha - \omega\gamma', \\
\dot{\alpha}'' &= \omega q - \alpha' p - \omega\gamma'',
\end{aligned} \tag{16.3}$$

$$\begin{aligned}
\dot{\beta} &= \beta' r - \beta'' q, \\
\dot{\beta}' &= \beta'' p - \beta r,
\end{aligned} \tag{16.4}$$

$$\begin{aligned}
\dot{\beta}'' &= \beta q - \beta' p, \\
\dot{\gamma} &= \gamma' r - \gamma'' q + \omega\alpha, \\
\dot{\gamma}' &= \gamma'' p - \gamma r + \omega\alpha', \\
\dot{\gamma}'' &= \gamma q - \gamma' p + \omega\alpha'',
\end{aligned} \tag{16.5}$$

Here A, P, C, are the principal moments of satellite inertia with reference to axes x', y', z', respectively; p, q, r, are the components of its instantaneous angular rotation velocity with respect to point 0; μ is the gravitational constant of the central field of the attraction forces of the Earth. $L_{x'}$, $L_{y'}$, $L_{z'}$ are moments of control, ω is the angular velocity of the satellite rotation over the orbit. It is known that /608

$$\omega = \frac{dv}{dt} = \frac{\sqrt{\mu P}}{R^2}, \quad R = \frac{P}{1+e\cos v}, \tag{16.6}$$

where P is the focal parameter of the orbit and e is its excentricity. For a circular orbit e = 0, R = P = const, $\omega^2 = \bar{\omega}^2 = \frac{\mu}{R^3} = \text{const}$. Equations (16.2) take

into account the effect of a central gravitational field and do not decompose for any initial perturbations. Equations (16.3) - (16.5) have six trivial integrals, establishing the relationships between the direction cosines, known from analytical geometry. In the case of a circular orbit (e = 0) with the controls turned off, Equations (16.2) - (16.5) admit a first additional nontrivial integral

$$\frac{1}{2}(Ap^2 + Bq^2 + Cr^2) + \frac{3}{2}\bar{\omega}^2(A\alpha^2 + B\beta^2 + C\gamma^2) - \bar{\omega}(A\alpha\alpha' + B\beta\beta' + C\gamma\gamma') = h, \tag{16.7}$$

where h is an arbitrary constant [16].

We will now introduce new variables into our analysis, i.e., the angular velocities $\bar{p}$, $\bar{q}$, $\bar{r}$ of satellite rotation about axes x', y', z'.

Then,

$$p = \bar{p} + \alpha\bar{\omega}, \quad q = \bar{q} + \beta\bar{\omega}, \quad r = \bar{r} + \gamma\bar{\omega}. \tag{16.8}$$

The position, in which we want to stabilize the satellite, is defined by equalities

$$\bar{p} = \bar{q} = \bar{r} = 0, \quad \alpha' = \beta' = \gamma' = 0. \tag{16.9}$$

These equalities have congruent stable values of the directional cosines

$$\alpha = \beta' = \gamma' = 1, \quad \alpha' = \alpha'' = \beta = \beta'' = \gamma = \gamma'' = 0. \tag{16.10}$$

The motion (16.9), (16.10) is interpreted as unperturbed motion; then, the variables $\bar{p}, \bar{q}, \bar{r}, 1+\Delta_1, \alpha', \alpha'', \beta, 1+\Delta_2, \beta'', \gamma, \gamma', 1+\Delta_3$ can be regarded as quantities, defining the perturbed motion of the satellite. These variables are governed by equations

$$
\begin{aligned}
A\dot{\bar{p}} &= \omega(B-C-A)\bar{r} + \omega^2(B-C)\beta'' + 3\omega^2(C-B)\gamma' + g_1 + u_1,\\
B\dot{\bar{q}} &= 3\omega^2(A-C)\gamma + g_2 + u_2,\\
C\dot{\bar{r}} &= \omega(A-B+C)\bar{p} + \omega^2(A-B)\beta + g_3 + u_3,\\
g_1 &= (B-C)\bar{q}\bar{r} + \omega(B-C+A)\bar{q}\beta'' + \omega(B-C-A)\bar{r}\Delta_2 +\\
&+ (B-C)\omega^2\beta''\Delta_2 + 3\omega^2(C-B)\gamma'\Delta_2 - A\dot{\omega}\beta,\\
g_2 &= (C-A)\bar{p}\bar{r} + \omega(C-A-B)\bar{p}\beta'' + \omega(C-A+B)\bar{r}\beta +\\
&+ (C-A)\omega^2\beta\beta'' + 3\omega^2(A-C)\gamma\Delta_2 - B\dot{\omega}(1+\Delta_2).
\end{aligned}
\tag{16.11}
$$

$$
\begin{aligned}
g_3 &= (A-B)\bar{p}\bar{q} + \omega(A-B-C)q\beta + \omega(A-B+C)p\Delta_2 +\\
&+ (A-B)\omega^2\beta\Delta_2 + 3\omega^2(B-A)\gamma\gamma' - C\dot{\omega}\beta'',\\
\dot{\Delta}_1 &= -\omega(\alpha'' + \gamma) + f_1,\\
\dot{\alpha}' &= -\bar{r} - \omega(\gamma' + \beta'') + f_2,\\
\dot{\alpha}'' &= \bar{r} + \omega(\Delta_1 + \Delta_2 - \Delta_3) + f_3,\\
\dot{\beta} &= \bar{r} + f_4,\\
\dot{\Delta}_2 &= f_5,\\
\dot{\beta}'' &= -\bar{p} + f_6,\\
\dot{\gamma} &= -\bar{q} + \omega(\Delta_1 - \Delta_2 - \Delta_3) + f_7,\\
\dot{\gamma}' &= \bar{p} + \omega(\beta + \alpha') + f_8,\\
\dot{\Delta}_3 &= \omega(\alpha'' + \gamma) + f_9,\\
f_1 &= \alpha'\bar{r} - \alpha''\bar{q} + \omega(\alpha'\beta'' - \alpha''\Delta_3),\\
f_2 &= \alpha''\bar{p} - \bar{r}\Delta_1 + \omega(\alpha''\beta - \beta''\Delta_1),\\
f_3 &= \bar{q}\Delta_1 - \bar{p}\alpha' + \omega(\Delta_1\Delta_3 - \alpha'\beta),\\
f_4 &= \bar{r}\Delta_2 - \beta''\bar{q};\ f_5 = \beta''\bar{p} - \beta\bar{r};\ f_6 = \beta\bar{q} - \bar{p}\Delta_2,\\
f_7 &= \gamma'(\omega\beta'' + \bar{r}) - \bar{q}\Delta_3 - \omega\Delta_2\Delta_3,\\
f_8 &= \bar{p}\Delta_3 + \omega(\beta\Delta_3 - \gamma\beta'') - \gamma\bar{r},\\
f_9 &= \gamma\bar{q} - \gamma'\bar{p} + \omega(\gamma\Delta_3 - \gamma'\beta).
\end{aligned}
\tag{16.12}
$$

When e = 0, all coefficients in Equations (16.11), (16.12), are constant; when /609
$0<e<1$, these coefficients are periodic functions of time; when e $\geq$ 1 all coefficients of those equations which include the multiplier ω are vanishing functions of time. Equations (16.11), (16.12) are precise, and the variables included in them can take on arbitrary values.

When e = 0, Equations (16.11), (16.12) have the following first integrals:

$$I_1 = \frac{1}{2}(A\bar{p}^2 + B\bar{q}^2 + C\bar{r}^2) + \frac{3}{2}\bar{\omega}^2[(A-C)\gamma^2 + (B-C)\gamma'^2] + \\ + \frac{1}{2}\bar{\omega}^2[(B-A)\beta^2 + (B-C)\beta''^2] = h_1, \tag{16.13}$$

$$I_2 = \Delta_1^2 + 2\Delta_1 + \alpha'^2 + \alpha''^2 = 0, \\ I_3 = \beta^2 + \Delta_2^2 + 2\Delta_2 + \beta''^2 = 0, \\ I_4 = \gamma^2 + \gamma'^2 + \Delta_3^2 + 2\Delta_3 = 0, \tag{16.14}$$

$$I_5 = (1+\Delta_1)\beta + \alpha'(1+\Delta_2) + \alpha''\beta'' = 0, \\ I_6 = (1+\Delta_1)\gamma + \alpha'\gamma' + \alpha''(1+\Delta_3) = 0, \\ I_7 = \beta\gamma + (1+\Delta_2)\gamma' + \beta''(1+\Delta_3) = 0. \tag{16.15}$$

Of these, the first one is obtained by transforming Integral (16.7) to new /610
variables, whereas the others are obtained by transforming trivial integrals of kinematic equations to the variables which define perturbed motion.

Consider the case of circular motion along an orbit. We introduce arbitrary positive constants $\kappa_1, \ldots \kappa_7$ and construct a Lyapunov function which is a linear combination of integrals (16.13) - (16.15):

$$V = \sum \kappa_s I_s. \tag{16.16}$$

When $A>C$, $B>A$ the function V is a fixed sign positive function with respect to all variables. In view of Equations (16.11) and (16.12) its total derivative has the form

$$\dot{V} = \bar{p}u_1 + \bar{q}u_2 + \bar{r}u_3 \tag{16.17}$$

We thus have a set M of feedback laws

$$\begin{aligned} u_1 &= -\bar{u}_1 \operatorname{sign} \bar{p}, \\ u_2 &= -\bar{u}_2 \operatorname{sign} \bar{q}, \\ u_3 &= -\bar{u}_3 \operatorname{sign} \bar{r} \end{aligned} \tag{16.18}$$

or

$$\begin{aligned} u_1 &= -\operatorname{sat}(\rho\bar{p}), \\ u_2 &= -\operatorname{sat}(\rho\bar{q}), \\ u_3 &= -\operatorname{sat}(\rho\bar{r}). \end{aligned} \tag{16.19}$$

We will demonstrate that both types of control guarantee the satellite the property of asymptotic stability of state determined by the solution of (16.9). Actually, with (16.18), we have

$$\dot{V} = -\bar{u}_1|\bar{p}| - \bar{u}_2|\bar{q}| - \bar{u}_3|\bar{r}|, \tag{16.20}$$

and an identical conversion of V to zero is possible only when $\bar{p} = \bar{q} = \bar{r} = 0$. However, then

$$p = \bar{a}\bar{\varphi}, \quad q = \bar{a}\bar{\psi}, \quad r = \bar{a}\bar{\gamma}, \tag{16.21}$$

and the Equations (16.2) will yield

$$\dot{\varphi} = \beta\gamma, \quad \dot{\psi} = \beta\gamma, \quad \dot{\gamma} = \beta\gamma, \tag{16.22}$$

which automatically leads to equalities $\varphi = \psi = \gamma = 0$. A similar conclusion follows also for (16.19).

Thus, the control of (16.18), (16.19) guarantees asymptotic stability. The closed system has the form

$$\begin{aligned}
A\dot{p} &= (B-C)qr + 3\bar{\omega}^2(C-B)\gamma'\gamma'' - \bar{u}_1 \operatorname{sign}(p-\alpha\dot{\beta}),\\
B\dot{q} &= (C-A)rp + 3\bar{\omega}^2(A-C)\gamma''\gamma - \bar{u}_2 \operatorname{sign}(q-\alpha\dot{\beta}'),\\
C\dot{r} &= (A-B)pq + 3\bar{\omega}^2(B-A)\gamma\gamma' - \bar{u}_3 \operatorname{sign}(r-\alpha\dot{\beta}'')
\end{aligned} \tag{16.23}$$

in conjunction with Equations (16.3) = (16.5).

Let show the control (16.18) also guarantees asymptotic stability in flight along any orbit if the constants $\bar{u}_1, \bar{u}_2, \bar{u}_3$ are sufficiently large.

In fact, when the satellite moves along any arbitrary orbit, the total derivative Function (16.16) computed on the strength of Equations (16.11), (16.12), is reduced to the form

$$\begin{aligned}
\dot{V} = &-\left[\bar{u}_1 - (\omega^2 - \bar{\omega}^2)\left[(B-C)(\beta''\beta' - 3\gamma''\gamma') - A\dot{\alpha}\beta\right]\right]|\bar{p}| -\\
&-\left[\bar{u}_2 - (\omega^2 - \bar{\omega}^2)\left[(C-A)(\beta''\beta - 3\gamma''\gamma) - B\dot{\alpha}\beta'\right]\right]|\bar{q}| -\\
&-\left[\bar{u}_3 - (\omega^2 - \bar{\omega}^2)\left[(A-B)(\beta\beta' - 3\gamma\gamma') - C\dot{\alpha}\beta''\right]\right]|\bar{r}|.
\end{aligned} \tag{16.24}$$

Asymptotic stability is guaranteed when the following additional conditions are maintained with reference to the control modules

$$\begin{aligned}
\bar{u}_1 &> \max_t |(\omega^2 - \bar{\omega}^2)\left[(B-C)(\beta''\beta' - 3\gamma''\gamma') - A\dot{\alpha}\beta\right]|,\\
\bar{u}_2 &> \max_t |(\omega^2 - \bar{\omega}^2)\left[(C-A)(\beta''\beta - 3\gamma''\gamma) - B\dot{\alpha}\beta'\right]|,\\
\bar{u}_3 &> \max_t |(\omega^2 - \bar{\omega}^2)\left[(A-B)(\beta\beta' - 3\gamma\gamma') - C\dot{\alpha}\beta''\right]|.
\end{aligned} \tag{16.25}$$

The question of whether or not it is possible, in the event of large deviations, to weaken the control field up to the state $P_{(p,2)}$ or $P_{(q,2)}$, remains unclear; in the event of small deviations, naturally, the linearization of Equations (16.11), (16.12) brings us to the results of Section 15.

PART IV

MIXED PROBLEMS OF OPTIMAL CONTROL THEORY

§17. Two Optimal Control Problems

The contemporary theory of optimal control considers two independant problems: the programming of optimal trajectories, and the problem of synthesis (analytical design) of feedback laws which make it possible for a given object to move along optimal trajectories. This classification emerged historically in the optimal control theory, because it was accepted without any reservation, going back to the period of classical control. It passed without any changes also into the theory of optimal control. The difference is in that at the present time the movement of x^*, u^* is, in a certain sense, optimal, and its selection is the first variational problem, solved by the theorem of the principle of the maximum. The optimal trajectory and the congruent motion of the controlling element u^* are determined as a known function of time and boundary conditions, stipulated in the framework of the problem.

Unfortunately, this motion, as a rule, is unstable with respect to arbitrarily small disturbances of the boundary conditions, i.e., the first problem is incorrect in A. N. Tikhonov's sense. Hence, considering the found trajecotry as a trajectory of unperturbed motion, we subsequently formulate the synthesis problem of feedback laws of ξ, which guarantee the stability for the found unperturbed motion. The synthesis problem is resolved by dynamic programming.

The groundlessness of such a division was pointed out in [17]. The truth of the matter is that the formulation of both problems uses the assumption that the control resources are strictly divided into two parts: the part that is required for the performance of motion u^*, and the part ξ,

required for the stabilization. Such a division, as if were, is of an arbitrary nature, and generally lies beyond the framework of the two variational problems of optimal control.

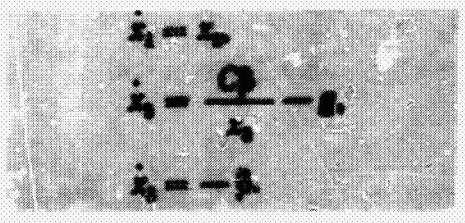
By performing an arbitrary separat n of the control resources, we can obtain, in lieu of unperturbed motion, programming trajectories which in a sense have many defects (a rather limited resource for u*); when the circumstances are adequate, we can obtain a stabilization (a sufficiently large reserve for ξ). The other alternative is in a different separation of the general control reserve; in this instance a good programmed motion of u* (in a sense) will be poorly stabilized in view of lacking control reserve of ξ. /612

A rejection of the arbitrary separatio: of the control reserve leads to mixed problems; their content and sense are demonstrated on the following example.

§18. The Moon Landing of a Spacecraft

Consider the moon landing problem of a spacecraft [18]. Assume that the ship is a material point exposed only to the influence of the forces of gravity of the central field, and to the reaction force.

Let us assume that the lunar landing takes place along the vertical of the landing site. Let x be the ship's coordinate; m - its mass, $c\beta$ - the reaction force, where β is the mass consumption per second, and c is a design constant. Introducing the notation $x = x_1, x_1 = x_2$, $m = x_3$, we obtain the equations of the mathematical model of the control object



(18.1)

In the conventional formulation of the programming problem, it is assumed that the control function β belongs to a set of piecewise continuous functions, which take on their values from the domain

$$|\beta| \leq \beta_* \tag{18.2}$$

We will temporarily retain this assumption. If we would have programmed a soft lunar landing trajectory, as we are doing now, from an arbitrary initial state

$$x_1(0) = x_{10}, \quad x_2(0) = x_{20}, \quad \tau_0(0) = x_{30} \tag{18.3}$$

into point

$$x_1(T) = x_{1T} = 0, \quad x_2(T) = x_{2T} = 0, \tag{18.4}$$

minimizing the fuel consumption

$$\Delta G = -x_3|_0^T = x_{30} - x_{3T}, \tag{18.5}$$

and if we would have used the entire reserve of control (18.2) (as is assumed in theory), we would have obtained a control program shown on Figure 2. It can be rendered as a known function of time t.

A soft landing program consists of two stages: AB, where $\beta = 0$ and the spacecraft falls freely during the time $t^* - 0$; BO, where $\beta = \beta_*$, i.e., the consumption is maximal providing for full braking of the spacecraft at point $x_{1T} = x_{2T} = 0$ during the time $T-t^*$. Here $f(x_1,x_2) = 0$, a known equation, is found by applying the principle of the maximum theorem, and moment t^* is the moment when the spacecraft enters into the path defined by this curve. In a superficial interpretation of the solution, it appears that the control can be performed automatically because knowing the equation

$f(x_1, x_2) = 0$ permits us to consider the control β in such a manner as if we would have been solving a synthesis problem from the very beginning. In fact, let us install two on-board measuring devices to obtain information on x_1, x_2. The readings of the devices, say, are fed into a computer which determines the numerical value of function $f(x_1, x_2)$, and let us begin the free fall from point A (Figure 3).

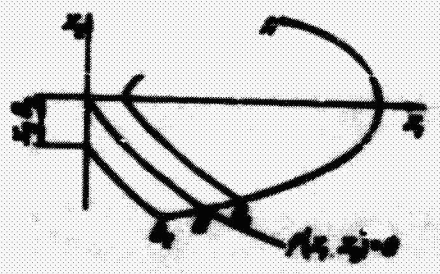


Figure 2. Soft lunar landing program

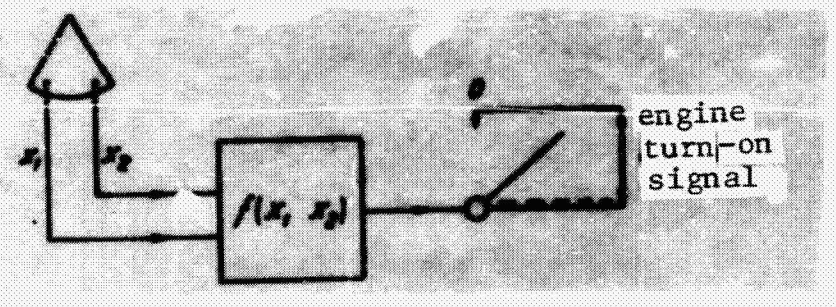


Figure 3. Automat for turning on engine.

Let variables z_1, z_2, z_3, u define the programmed motion of lunar landing, and variables y_1, y_2, y_3, ξ define the perturbed motion. We have

$$\begin{aligned} x_1 &= z_1 + y_1, \\ x_2 &= z_2 + y_2, \\ x_3 &= z_3 + y_3, \quad \beta = u + \xi. \end{aligned} \tag{18.6}$$

The system of variables z, u satisfies equations

$$\begin{aligned} \dot{z}_1 &= z_2, \\ \dot{z}_2 &= \frac{cu}{z_3} - g, \\ \dot{z}_3 &= -u \end{aligned} \tag{18.7}$$

and boundary conditions /614

$$(i) \quad z(0) = z_0, \qquad (j) \quad z_1(T) = z_2(T) = 0, \tag{18.8}$$

T is arbitrary

Programmed control u must be so selected as to minimize the fuel consumption

$$\Delta G = -z_0|_0^T = z_{00} - z_{0T} = \min. \tag{18.9}$$

For variables y,ξ we have equations of perturbed motion (equations in first approximation)

$$\begin{aligned} \dot{y}_1 &= y_2, \\ \dot{y}_2 &= \frac{\alpha_1}{z_0^2} y_3 + \frac{c}{z_0} \xi, \\ \dot{y}_3 &= -\xi. \end{aligned} \tag{18.10}$$

The initial values of y_0 can be arbitrary, belonging to the domain $\alpha_1 y_{10}^2 + \alpha_2 y_{20}^2 + \alpha_3 y_{30}^2 \leq R^2$, where $\alpha_1, \alpha_2, \alpha_3, R$ are given integers. Equations (18.7), (18.10) are not independent and controls u,ξ are connected by the total reserve

$$0 \leq u + \xi \leq \beta_2 \tag{18.11}$$

In addition, generally speaking, it does not follow that System (18.10) will be controllable for all programs z, u determined by relations (18.7) - (19.8), (18.11). Thus, the coefficients of Equations (18.10) must satisfy the rule of the general position in order to have any meaning. In this instance, this rule (sufficient condition of controllability) has the form [9]

$$A(t)=\begin{vmatrix}0 & 1 & 0\\ 0 & 0 & \frac{a_1}{z_0}\\ 0 & 0 & 0\end{vmatrix},\quad B(t)=\begin{vmatrix}0\\ \frac{c}{z_0}\\ -1\end{vmatrix}, \tag{18.12}$$

$$L_1=B;\quad L_2=AL_1-\frac{dL_1}{dt},\quad L_3=AL_2-\frac{dL_2}{dt},$$

range $(L_1, L_2, L_3)=3$.

One of the substantial restrictions, stipulated by Rule (18.12), is that now the optimal controls of u cannot be piecewise continuous functions. To complete the formulation of the problem, it is necessary to introduce a functional which estimates the measure of deviation of the actual trajectory from the programmed trajectory. We are dealing here with a terminal control problem, and it makes sense to estimate the quality of a soft lunar landing by the functional

$$\Delta G_3=\beta_{1T}^2+\alpha\beta_{2T}^2+\beta\beta_{3T}^2, \tag{18.13}$$

where the weight coefficients $\alpha>0, \beta>0$. Hence in the new variant, the optimal control problem (if this is a problem?) is a mixed one, and must be formulated as follows.

Given: Equations (18.7), (18.10), boundary conditions (18.8) for variable z, range of initial disturbances for variable y, Functional (18.9), estimating the programmed control of u, Functional (18.13), estimating the stabilization quality with the feedback law $\xi(t, n)$. Admissible controls of ξ are a set of piecewise continuous functions, and the admissible program u of lunar landing are a set of piecewise smooth functions, satisfying restrictions (18.7) and taking on the n, values from domain (2.11). The problem is to determine the optimum lunar landing program and optimal feedback law. The formulated program does not include alternatives for separating a unified control source into two parts. It is conceivable that this formulation will be supplemented by the requirement that with the given optimal control program the feedback law would guarantee in a closed system (18.10) the required dimensions of the gravitation region. /615

§19. Other Forms of Boundary Value Problems

Other forms of boundary value problems arise when we refuse to search for the optimal feedback law. Instead, we only stipulate the conditions that arbitrarily arising errors y_0 would decrease in time, and that the quantity ΔQ_δ at moment $t = T$ be sufficiently small. Sample formulations are included in the first two parts of this study. Consider the simplest one of them.

Given: Equations (18.7) (18.10), boundary conditions (18.8) for variable z, domain of initial perturbations for variable y, Functional (18.9) which estimates the control program, and Functional (18.13) for estimating stabilization quality. The admissible controls of ξ are a set of piecewise continuous functions, whereas the admissible programs are a set of piecewise smooth functions, taking on their values from Domain (2.11), (2.12).

Let δ be a given positive integer.

Problem: Let us define the optimum landing program and set of feedback laws guaranteeing such a decrease of $y(t)$ $1<t<T$, that condition

$$\Delta Q_\delta < \delta \tag{19.1}$$

is satisfied.

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